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A streamline coordinate system for distorted two-dimensional shear flows

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A function ϕ is derived which is constant along the orthogonal trajectories of streamlines in two-dimensional flow. In irrotational flows, ϕ reduces to the velocity potential. The pair of functions ϕ and ψ, where ψ is the stream function, are used to define a coordinate system in rotational fluid flows. Tensor methods are used to transform the equations of motion of a turbulent fluid and the equations for second moments of turbulent fluctuations to this coordinate system. Explicit extra terms appear in the transformed equations embodying the effects of streamline curvature and mean flow acceleration. These extra terms are characterized by two lengthscales which arise naturally from the transformation: the local radius of curvature of the streamline and the 'e-folding' distance of the mean streamwise velocity.

1. Introduction

The investigation of distorted shear flows, flows with streamline curvature and acceleration, leads to some of the most difficult problems of aerodynamics and engineering fluid mechanics. Recently, micrometeorologists have begun to make systematic attacks on the related field of flow over 'complex terrain'. A primary difficulty in every case is to choose a coordinate frame that simplifies both the correlation of measured data and the construction of predictive models. The rectangular Cartesian coordinate system is not usually appropriate since the physical interpretation of many quantities (for instance turbulent Reynolds stresses) becomes elusive when flow direction and coordinate direction do not coincide. To estimate changes in properties of the flow between two points on a streamline requires integration along a curve, generally a complicated operation, and, at least in micrometeorological field measurements, there are practical difficulties in aligning instruments accurately with some notional, externally imposed rectangular frame.

The obvious recourse has been to curvilinear coordinates. If one coordinate direction can be chosen almost parallel to the mean flow direction, then extra terms arising from deviation of the mean flow from the coordinates may be small enough to be approximated in calculation schemes or ignored in interpretation of measurements. Bradshaw (1973) reviewed the coordinate systems appropriate to different classes of distorted shear flows and suggested the use of '(s, n)' coordinates for two-dimensional thin shear layers. The (s, n)-system was developed in detail by Howarth (1951) and is summarized in figure 1.

The (s, n)-system has the disadvantage that, except for simple rotating flows where R, the radius of curvature of the s-coordinate lines, is not a function of distance along them (we include the 'flat' case $R = \infty$), the coordinate lines can be parallel to only one streamline. Elsewhere there will be mean velocity components and advection

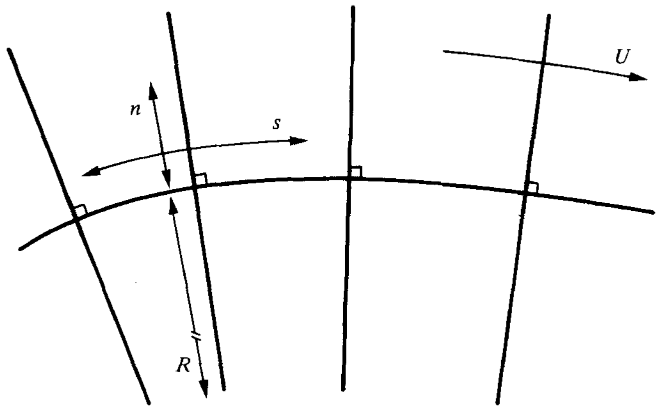

FIGURE 1. The (s, n)-coordinate system.

perpendicular to the s-lines. This is not a serious limitation in thin shear layers, where the direction of the s-lines can always be chosen so that they correspond closely to the flow direction, but this is not possible in strongly distorted flows; in this case the added complexity of the (s, n)-system brings with it no advantages.

A more serious limitation may be the failure of the transform at the locus of the radii of curvature of the s-lines where the Jacobian of the transform becomes singular. For high curvature (such as may occur in flows around bluff bodies) this locus may be within the region of interest or domain of intended calculation. A final unsatisfactory feature of the system is the necessity of supplying the coordinate curves *a priori*, so that some of the global properties of the flow field must be known before the coordinates can be established. For attached boundary layers over slightly curved walls this is not really a problem since the surface provides an adequate descriptor; thin free shear layers are not so well provided.

Most of these problems would disappear if streamlines could be used as coordinate lines. In irrotational flows, conformal mapping provides a streamline-orthogonal trajectory net, although the further step of transforming the equations of fluid flow into these coordinates is rarely taken. However, Lighthill (1956) did essentially this in his investigation of irrotational distortion of vorticity convected past a sphere, while Durbin & Hunt (1980) demonstrated the simplicity to be achieved when Hunt's rapid-distortion theory is developed in streamline coordinates. In rotational flows, unfortunately, the velocity potential ϕ is not defined and the methods of conformal transformation are not available to us. This paper is an attempt to remedy this deficiency.

In the sections that follow we first develop the rotational analogue of ϕ and show how it may be combined with the stream function to form a useful coordinate system. In §3, the formalism of tensor analysis will be employed to derive the equations of turbulent fluid motion in these coordinates, and in §4 the resulting first- and second-moment equations will be presented and discussed. Finally, in §5 we attempt to clarify some points that might arise in formulating mathematical models in the coordinate system.

2. The coordinate frame

A practical system of coordinate lines in a three-dimensional Euclidian space consists of the intersections of surfaces x_i described in implicit form, i.e.

$$x_i(y_1, y_2, y_3) = \text{constant}, \tag{2.1}$$

where y_i are the components of a position vector in a rectangular Cartesian reference framework. In order to obtain an orthogonal streamline coordinate system the x_i must comprise two orthogonal stream surfaces and the surface normal to them. It can be shown (e.g. Piaggio 1958, p. 140) that such a *normal congruence* of surfaces only exists if the velocity field is *complex lamellar*, that is, if $\boldsymbol{v} \cdot \operatorname{curl} \boldsymbol{v} = 0$, where $\boldsymbol{v}$ is the velocity vector. Two-dimensional and axially symmetric flows form members of this class, and in the analysis that follows we will restrict ourselves to the two-dimensional case. The three surfaces x_i are therefore

$$x_3 = y_3 = \text{constant}, \quad \text{the plane of symmetry,}$$

$$x_2 = \psi(y_1, y_2) = \text{constant}, \quad \text{the stream function,}$$

$$x_1 = \phi(y_1, y_2) = \text{constant}, \quad \text{the surface orthogonal to the } \psi \text{ and } x_3 \text{ surfaces.}$$

The streamlines are the vector lines of the velocity field, or, since we are working in two dimensions, the lines $\psi = \text{constant}$. The stream function ψ has the usual properties, viz

$$\psi - \psi_0 = \int (V_1 \, \mathrm{d}y_2 - V_2 \, \mathrm{d}y_1), \tag{2.2}$$

$$V_1 = \frac{\partial \psi}{\partial y_2}, \quad V_2 = -\frac{\partial \psi}{\partial y_1}, \tag{2.3}$$

where V_1 and V_2 are the components of mean velocity in the directions y_1 and y_2 respectively.

The orthogonal trajectories to the streamlines are the lines $\phi = \text{constant}$; their tangents are proportional to $\boldsymbol{\nabla}\psi$ so they must satisfy the o.d.e.

$$V_1 \, \mathrm{d}y_1 + V_2 \, \mathrm{d}y_2 = 0. \tag{2.4}$$

Since, as we have already pointed out, a solution to (2.4) exists in complex-lamellar flows, it is possible to find an integrating factor $\zeta(y_1, y_2)$ such that

$$(\zeta V_1) \, \mathrm{d}y_1 + (\zeta V_2) \, \mathrm{d}y_2 = 0 = \mathrm{d}\phi \tag{2.5}$$

is an exact differential and

$$\frac{\partial \phi}{\partial y_1} = \zeta V_1, \quad \frac{\partial \phi}{\partial y_2} = \zeta V_2. \tag{2.6}$$

A coordinate transform is completely defined by its metric, a quantity which contains only the partial differential coefficients of the coordinate surfaces x_i with respect to the reference framework, in this case the Cartesian coordinates y_i. We will consider these coefficients completely specified if they can be written in terms of Cartesian velocity components, and it is therefore clear from (2.3) and (2.6) that the problem becomes one of specifying the integrating factor ζ.

This can be done most directly by recognizing that in two dimensions ψ is the only non-zero component of the vector potential $\boldsymbol{\psi} = (0, 0, \psi)$. The orthogonality condition can then be written:

$$\boldsymbol{\nabla}\phi = \zeta \boldsymbol{\nabla} \times \boldsymbol{\psi}, \tag{2.7}$$

and a constraint on ζ is obtained by taking the curl:

$$0 = \boldsymbol{\nabla} \times (\zeta \boldsymbol{\nabla} \times \boldsymbol{\psi}) = (0, 0, \boldsymbol{\nabla}\zeta \cdot \boldsymbol{\nabla}\psi + \zeta \nabla^2 \psi), \tag{2.8}$$

or

$$0 = |\boldsymbol{\nabla}\psi|^2 \frac{\partial \zeta}{\partial \psi} + \zeta \nabla^2 \psi. \tag{2.9}$$

Since $|\nabla\psi|^2 = V_1^2+V_2^2 = Q^2$ and $\nabla^2\psi = \partial V_1/\partial y_2 - \partial V_2/\partial y_1 = -\Omega$, where Ω is the vorticity (this definition of Ω is appropriate to a right-handed coordinate system, a convention we will retain throughout), (2.9) can be written

$$\frac{\Omega}{Q^2} = \frac{\partial \ln \zeta}{\partial \psi}. \tag{2.10}$$

In irrotational flows $\Omega = 0$, and ζ is a constant, which without loss of generality may be set equal to 1. We see from (2.6) that ϕ then becomes the familiar potential function of hydrodynamics. When the vorticity is non-zero

$$\frac{\zeta}{\zeta_0} = \exp \int \frac{\Omega}{Q^2}\, d\psi, \tag{2.11}$$

and it is clear from (2.5) that we may choose $\zeta_0 = 1$.†

In general, complete knowledge of the stream function is necessary to compute ζ. It will become apparent in §3, however, that, when the streamlines are steady, we will only require knowledge of $\partial \ln \zeta/\partial\psi|_{\phi=\text{const},\, x_3=\text{const}}$, the result (2.10).

An alternative procedure can be adopted to obtain ζ. If we apply the integrability condition

$$\frac{\partial^2\phi}{\partial y_1 \partial y_2} = \frac{\partial^2\phi}{\partial y_2 \partial y_1}$$

to (2.6), we obtain a first-order p.d.e. for ζ which can be solved by the method of characteristics, the characteristic curves being simply the orthogonal trajectories. This method is, in fact, more general than that described above, which relies on the particular relationship between vector potential and stream function in two-dimensional flow. It can be used, for example, in the analogous problem that arises in specifying the density of sea water in terms of its temperature and salinity. The density obeys Poisson's equation, the right-hand side of which can only be expressed in explicit form by solving an orthogonality problem of the type considered here. Veronis (1972) and Mamayev (1973) discuss this problem, and proceed by replacing Poisson's equation by Laplace's equation. The analysis outlined here, however, allows an exact solution to be obtained.

3. Transforming the turbulence equations into the new coordinate frame

We employ the techniques of general tensor analysis to derive the equations in the new system. Readers unfamiliar with this formalism may skip this section and go directly to the discussion of the transformed equations in §4.

A good textbook on this subject, which has a particular orientation towards fluid mechanics, is Aris (1962), while Bradshaw (1973, appendix I) presents a concise introduction to the use of general tensors in turbulent flows. In this section only, we employ the convention that superscripts denote contravariant tensor components and subscripts covariant components. The summation convention operates only between indices of different variance, hence if u^i_j is a second-order tensor of mixed variance, u^i_i is its trace, a scalar, while u^{ii} and u_{ii} are second-order doubly contravariant and covariant tensors respectively. On the occasions when a superscript denotes an exponent, the meaning will be obvious from the context.

† The author would like to acknowledge one of his referees, who persuaded him to adopt this succinct derivation of ζ.

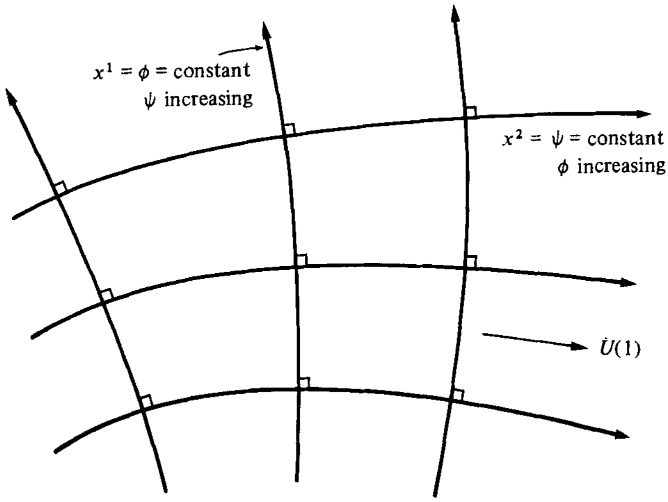

FIGURE 2. The (ϕ, ψ, x^3)-coordinate system.

We denote the contravariant components of the new coordinates by x^i, where

$$x^1 = \phi, \quad x^2 = \psi, \quad x^3 = y^3. \tag{3.1}$$

The coordinates are shown schematically in figure 2.

The contravariant metric tensor g^{pq} is given by

$$g^{pq} = \sum_i \frac{\partial x^p}{\partial y^i}\frac{\partial x^q}{\partial y^i} = \begin{pmatrix} \zeta^2 Q^2 & 0 & 0 \\ 0 & Q^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.2}$$

The x^i system is orthogonal, so the covariant metric is

$$g_{pq} = (g^{pq})^{-1} = \begin{pmatrix} \zeta^{-2} Q^{-2} & 0 & 0 \\ 0 & Q^{-2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.3}$$

The Jacobian of the transform is denoted by J, and

$$J = \begin{vmatrix} \zeta V_1 & \zeta V_2 & 0 \\ -V_2 & V_1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = g^{\frac{1}{2}}, \tag{3.4a}$$

where

$$g = |g^{ij}|. \tag{3.4b}$$

It is clear from (3.4a) that, when $Q = 0$, $J = 0$, so that stagnation points and solid surfaces (where $Q = 0$ is ensured by the no-slip condition) are excluded from the manifold of the transform. This is the natural extension of the singularities that appear in conformal mappings at stagnation points.

The covariant derivative with respect to the ith contravariant coordinate is denoted by $_{,i}$. Hence if u^i is a contravariant vector

$$u^i_{,j} = \frac{\partial u^i}{\partial x^j} + \Gamma^i_{kj} u^k, \tag{3.5}$$

where Γ^i_{jk} is the Christoffel symbol of the second kind and is defined by

$$\Gamma^i_{jk} = \frac{\partial y^i}{\partial x^m}\frac{\partial^2 x^m}{\partial y^j\,\partial y^k} \qquad (3.6a)$$

or

$$\Gamma^i_{jk} = \tfrac{1}{2}g^{ip}\left[\frac{\partial g_{pj}}{\partial x^k}+\frac{\partial g_{pk}}{\partial x^j}-\frac{\partial g_{jk}}{\partial x^p}\right]. \qquad (3.6b)$$

The Γs for the present metric are set out in table 1.

$$\Gamma^1_{11} = \zeta Q\frac{\partial}{\partial x^1}\left(\frac{1}{\zeta Q}\right),\quad \Gamma^2_{22} = Q\frac{\partial}{\partial x^2}\left(\frac{1}{Q}\right),\quad \Gamma^3_{33} = 0$$

$$\Gamma^1_{21} = \Gamma^1_{12} = \zeta Q\frac{\partial}{\partial x^2}\left(\frac{1}{\zeta Q}\right) = -\frac{1}{Q^2}\left(\Omega+\frac{\partial Q}{\partial x(2)}\right),\quad \Gamma^1_{31} = \Gamma^1_{13} = 0$$

$$\Gamma^2_{12} = \Gamma^2_{21} = Q\frac{\partial}{\partial x^1}\left(\frac{1}{Q}\right),\quad \Gamma^2_{23} = \Gamma^2_{32} = 0$$

$$\Gamma^2_{11} = -\frac{Q}{\zeta}\frac{\partial}{\partial x^2}\left(\frac{1}{\zeta Q}\right) = \frac{1}{\zeta^2 Q^2}\left(\Omega+\frac{\partial Q}{\partial x(2)}\right)$$

$$\Gamma^1_{22} = -\zeta^2 Q\frac{\partial}{\partial x^1}\left(\frac{1}{Q}\right) = \frac{\zeta}{Q^2}\frac{\partial Q}{\partial x(1)}$$

$$\Gamma^1_{33} = \Gamma^3_{11} = 0$$

TABLE 1. Christoffel symbols for the metric

$$g_{ij} = \begin{pmatrix} \zeta^{-2}Q^{-2} & 0 & 0 \\ 0 & Q^{-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The coordinates x^i do not all have dimensions of length, their dimensions being respectively L^2/T, L^2/T, L. Vectors representing familiar quantities will therefore acquire unfamiliar dimensions when referred to these coordinates. For example, the velocity vector u^i is the time derivate of the position vector of a point, that is, $u^i = \mathrm{d}x^i/\mathrm{d}t$, so that the contravariant components of the velocity vector u^1, u^2, u^3 have dimensions L^2/T^2, L^2/T^2, L/T respectively. Tensors referred to the x^i system may be regarded as having their components aligned with a triad of orthogonal base vectors, the tangent vectors of the coordinate lines, which have dimensions T/L, T/L, 0 and which change both their magnitude and direction from place to place when compared with the reference Cartesian system. If we normalize the vectors of the basis by their local magnitudes, we acquire an orthonormal basis which merely varies in orientation from place to place. Tensors referred to this system have familiar dimensions and in fact correspond to measurable quantities. For example, the contravariant components u^i of the velocity vector would be replaced by the 'physical' components' $u(i)$, all of which have dimensions L/T and which are just the components we would measure in a rectangular Cartesian frame, locally tangent to the streamlines. To be of any practical use, the transformed equations must be expressed in terms of physical components. As we shall see below, rewriting the equations in terms of physical components has the further important consequence of removing singularities at stagnation points and solid surfaces. The reader is referred to appendix A for a more rigorous treatment of these points and to Truesdell

(1953) for an illuminating discussion of the relationship between tensors and their physical components.

Once the metric tensor has been specified, the Navier–Stokes equations in the new system can be obtained by substituting the components of the metric into the invariant forms of these equations. In the case of a turbulent flow, however, considerable labour is involved, particularly in deriving the equations of Reynolds stress components so that it was thought useful to present these equations here once and for all. A geometrical interpretation of some of the terms which appear in the transformed equations is also valuable and these will be the concerns of the rest of this paper.

The procedure adopted in deriving the transformed equations follows Bradshaw (1973, appendix I). It is to

(1) write down the original equation in Cartesian form;

(2) rewrite the equation in general tensor form, replacing partial derivatives by covariant derivatives and ensuring that all terms have the same variance;

(3) substitute for the covariant derivatives by (3.5);

(4) recover physical components.

The method is illustrated by the derivation of the streamwise mean momentum equation of a turbulent fluid flow.

In Cartesian coordinates we have

$$\frac{\partial V^1}{\partial t}+V^j\frac{\partial V^1}{\partial y^j}=-\frac{1}{\rho_0}\frac{\partial P}{\partial y^1}-\frac{\partial \overline{v^1v^j}}{\partial y^j}+\nu\frac{\partial^2 V^1}{\partial y^j\,\partial y^j}+\frac{g^1\Delta\rho}{\rho_0}. \tag{3.7}$$

From here on, capital letters denote the ensemble-mean part of a fluctuating quantity and small letters the deviation from the mean, so that U^i+u^i is the total velocity vector. The overbar $\overline{}$ denotes the averaging operation. g^1 is the component of the acceleration due to gravity in the y^1 direction, $\Delta\rho$ the deviation of the density from the hydrostatic value ρ_0, and ν the kinematic viscosity. Equation (3.7) is correct to the Boussinesq approximation, and Coriolis terms have been neglected (see, however, the appendix). Rewriting (3.7) in general tensor form:

$$\frac{\partial U^1}{\partial t}+U^jU^1_{,j}=-\frac{g^{1j}}{\rho_0}P_{,j}-(\overline{u^1u^j})_{,j}-\nu g^{jk}U^1_{,jk}+g^1\frac{\Delta\rho}{\rho_0}, \tag{3.8}$$

where U^i is the velocity in the x^i coordinate system.

We next substitute for the covariant derivatives by (3.5) and, noting that $U^2=U^3=0$, obtain

$$\frac{\partial U^1}{\partial t}+U^1\frac{\partial U^1}{\partial x^1}+U^1\Gamma^1_{11}=-\frac{g^{11}}{\rho_0}\frac{\partial P}{\partial x^1}-\frac{\partial\overline{u^1u^j}}{\partial x^j}-\Gamma^j_{hj}(\overline{u^1h^h})$$
$$-\Gamma^1_{hj}(\overline{u^hu^j})+\text{viscous terms}+\frac{g^1}{\rho_0}\Delta\rho. \tag{3.9}$$

(We will not write the viscous terms out explicitly in this illustrative example. A large number of terms are involved and their expression does not clarify the method.)

The relationship between the contravariant components of a tensor u^i and its physical components $u(i)$ in an orthogonal coordinate system is simply

$$u^i=\frac{u(i)}{g^{\frac{1}{2}}_{ii}}\quad\text{(no summation)}. \tag{3.10}$$

This relationship is given a physical basis in the appendix.

Substituting for the contravariant components in (3.9) after first replacing the Christoffel symbols by the values given in table 1 and invoking symmetry about the x^3 = constant plane leads to

$$\frac{\partial U(1)}{\partial t}+U(1)\frac{\partial U(1)}{\partial x(1)}+U(1)\frac{\partial \ln(\zeta Q)}{\partial t}$$
$$=-\frac{1}{\rho_0}\frac{\partial P}{\partial x(1)}-\frac{\partial}{\partial x(1)}(\overline{u(1)u(1)})-\frac{\partial}{\partial x(2)}(\overline{u(1)u(2)})+(\overline{u(1)u(1)}-\overline{u(2)u(2)})\frac{1}{Q}\frac{\partial Q}{\partial x(1)}$$
$$+\overline{2u(1)u(2)}\frac{1}{Q}\left[\Omega+\frac{\partial Q}{\partial x(2)}\right]+\frac{g(1)}{\rho_0}\Delta\rho+\text{viscous terms.} \quad (3.11)$$

We note that the integrating factor ζ does not appear in the final form of the equation except in the third term on the left-hand side, which vanishes in steady flows. As the reader may easily verify, ζ enters first through the Christoffel symbols when the covariant derivatives are expanded and again through components of the metric (scale factors) when contravariant components are replaced by their physical equivalents ((3.10)). When terms in ζ with opposite sign resulting from these two causes are cancelled, ζ remains only in the form

$$\frac{1}{\zeta}\frac{\partial}{\partial x^2}\zeta=\frac{\partial}{\partial x^2}\ln\zeta.$$

However $(\partial/\partial x^2)\ln\zeta$ is simply $(\partial/\partial\psi)\ln\zeta|_{\phi,x^3}$ in the y^i Cartesian coordinates, and, as we noted in (2.10), $(\partial/\partial\psi)\ln\zeta=\Omega/Q^2$. The result is the disappearance of ζ from the transformed equations.

The equation for the mean momentum balance normal to a streamline follows in the same way as (3.11):

$$\frac{U(1)^2}{Q}\left(\Omega+\frac{\partial Q}{\partial x(2)}\right)=-\frac{1}{\rho_0}\frac{\partial P}{\partial x(2)}-\frac{\partial}{\partial x(1)}(\overline{u(1)u(2)})-\frac{\partial}{\partial x(2)}(\overline{u(2)u(2)})$$
$$+\overline{2u(1)u(2)}\frac{1}{Q}\frac{\partial Q}{\partial x(1)}+[\overline{u(2)u(2)}-\overline{u(1)u(1)}]\frac{1}{Q}\left[\Omega+\frac{\partial Q}{\partial x(2)}\right]$$
$$+\frac{g(2)}{\rho_0}\Delta\rho+\text{viscous terms.} \quad (3.12)$$

Here $Q^2=V_1^2+V_2^2$, where V_1 and V_2 are the components of the mean velocity in the Cartesian coordinates y^i. However, by definition of a physical component, $U(1)^2=Q^2$. The x^1 coordinate is positive in the direction of increasing ϕ, which in turn is given by the expression

$$\phi-\phi_0=\oint[(\zeta V_1)\,\mathrm{d}y_1+(\zeta V_2)\,\mathrm{d}y_2].$$

From (2.11), ζ is positive definite. V_1 and V_2 are positive in the direction of positive y_1 and y_2, so that, along a streamline, x^1 increases in the direction of the mean velocity vector. In recovering physical quantities through (3.10), we choose the positive square root of g_{ii} so that the positive direction of the velocity vector defines the positive direction of $x(1)$. It follows that $U(1)$ is always positive and $U(1)=Q$. The positive $x(2)$ direction is in the direction of increasing stream function, and the specification of the handedness of the coordinate system is completed by the definition of the vorticity.

The new coordinate system is defined by using the mean vorticity, Ω. This means that any attempt to write Ω in terms of the transformed velocities leads either to an expression containing ζ or to the trivial result $\Omega' = \Omega$, where Ω' is the form of Ω in the x^i system. Ω is an invariant of the transform. If the velocity field is known in the x^i coordinate system, knowledge of Ω' is necessary for the transformation back to Cartesian coordinates. Ω' in this case should be calculated from the expression $\Omega = \operatorname{curl} U^i$, an expression which takes a particularly simple form in terms of the geometry of the flow field as we shall see below. We recognize that Ω is the third component of a contravariant vector, but, because of its special status as an invariant of the transform and because in two-dimensional flow it is the only non-zero component of the mean vorticity vector, we have continued to write Ω rather than Ω^3.

A useful result follows from transforming the incompressible continuity equation for mean velocities. In tensor form this is simply

$$U^1_{,1} + U^2_{,2} = 0. \tag{3.13}$$

Replacing the covariant derivatives by (3.5) and then recovering physical quantities leads to

$$\left.\begin{aligned} U^1_{,1} \to \frac{\partial U(1)}{\partial x(1)}, \quad U^2_{,2} \to -\frac{\partial U(1)}{\partial x(1)}, \\ U^1_{,1} + U^2_{,2} \to \frac{\partial U(1)}{\partial x(1)} - \frac{\partial U(1)}{\partial x(1)} = 0. \end{aligned}\right\} \tag{3.14}$$

Inspection of (3.11) and (3.12) shows that some simplification has been achieved in the advective terms – advection now only occurs along a streamline, the $x(1)$ direction, at the expense of acquiring additional higher moment terms. These take the form of Reynolds stresses divided by lengthscales, where the lengthscales are

$$\left(\frac{1}{U(1)}\frac{\partial U(1)}{\partial x(1)}\right)^{-1}, \quad \left(\frac{1}{U(1)}\left(\Omega + \frac{\partial U(1)}{\partial x(2)}\right)\right)^{-1}$$

(we have replaced Q by $U(1)$). The first lengthscale we denote by L_a and we can write either

$$\frac{\partial}{\partial x(1)} \ln U(1) = \frac{1}{L_a} \tag{3.15a}$$

or

$$\frac{U(1)}{L_a} = \frac{\partial U(1)}{\partial x(1)}. \tag{3.15b}$$

In other words, L_a is the 'e-folding' distance of the streamwise velocity and is the natural lengthscale of streamwise accelerations. A geometric interpretation of the second lengthscale follows from (3.12). The left-hand side of this equation is the total acceleration normal to a streamline, or simply the centripetal acceleration of a fluid particle, and may be rewritten as $U(1)^2/R$, where R is the local radius of curvature of the streamline. Comparing the two expressions for the acceleration we see that

$$\frac{1}{R} = \frac{1}{U(1)}\left(\Omega + \frac{\partial U(1)}{\partial x(2)}\right). \tag{3.16}$$

It is apparent from (3.16) that the sign of R is not now a matter of convention but depends upon the sign of the vorticity. In fact, if the local centre of curvature lies in the direction of increasing $x(2)$, R will be positive and vice versa. A change to a left-handed coordinate system inverts the sign of Ω and consequently of R.

Recovering physical components of vectors and tensors by application of (3.10) is equivalent to normalizing the vector basis. The new orthonormal basis is the triad of vectors consisting of the unit tangent to the streamline (the $x(1)$ direction), the unit principal normal (the $x(2)$ direction) and the unit binormal (the $x(3)$ direction). We can denote these three Cartesian vectors by $\boldsymbol{g}(i)$. The metric tensor of a transform can be constructed from the scalar products of the base vectors through the formula

$$g_{ij} = \boldsymbol{g}(i)\cdot\boldsymbol{g}(j) \tag{3.17}$$

(Aris 1962, p. 163). The metric of the complete transform, that is, the transformation into x^i coordinates followed by the recovery of physical coordinates, is therefore

$$g'_{ij} = g'^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{3.18}$$

The Jacobian of the complete transform is

$$J' = |g'^{ij}|^{\frac{1}{2}} = 1, \tag{3.19}$$

and the singularities that surrounded the initial transform domain have disappeared. A more physical feeling for this process may be obtained if it is appreciated that the original singularities correspond to points where the dimensional basis attains zero magnitude. Non-dimensionalizing the basis then has an obvious effect.

In (3.4) we have included the term $\partial U(1)/\partial t$, describing the temporal change of streamwise mean momentum, the mean, of course, being an ensemble mean. It should be apparent that a change in $U(1)$ must be accompanied by a change in the mean streamline pattern, that is, in the coordinate frame. Referring velocities to a coordinate frame in general motion results in an extra 'apparent' body force, the third term in (3.4), $U(1)(\partial/\partial t)\ln(\zeta Q)$. This term is analogous to the Coriolis 'force' that appears when the momentum equation is referred to coordinates undergoing rigid-body rotation, and indeed includes any such 'Coriolis' force. The form of this extra term, which has counterparts in the second-moment equations, is discussed in detail in the appendix, where we show that the Coriolis forces may be absorbed in this term.

In almost all practical cases, however, it makes no sense to refer velocities to a fluctuating coordinate frame. An exception is the case of a wave propagating through a turbulent field, when it is desirable to separate random turbulent fluctuations from coherent wave fluctuations, the latter being regarded as a part of the background flow (see e.g. Finnigan & Einaudi 1981). Even in this case, if the wave phase velocity c^i is constant in space, the streamline pattern may be frozen by a Galilean transformation of axes $\bar{y}^i = y^i - c^i t$ before the transformation to curvilinear axes. If c^i is a function of y^i, however, or if the wave field takes the form of a wave 'packet', no single Galilean transform will freeze the streamlines. In the second-moment equations that follow it should be recognized that a temporal change of Reynolds stresses will usually be accompanied by a change in $U(i)$.

4. The equations of first and second moments of velocity

Since the three coordinate directions are now asymmetric, we must write out the equations for each velocity or Reynolds-stress component separately. The equations are clarified by making a further (and final) change in notation. We replace $x(i)$ by x, y, z, where x, y, z represent actual distances along the streamline, orthogonal trajectory and x^3 lines respectively. Similarly, $U+u$, $V+v$, $W+w$ are the mean plus fluctuating velocity components in the x-, y-, z-directions. As mentioned in §3, these are velocities that would be measured by an anemometer in a Cartesian frame with its x-axis locally tangent to the streamline. g_x and g_y are the components of the acceleration due to gravity in the x-and y-directions, and $g_z = 0$.

The mean momentum equations are

$$\frac{\partial U}{\partial t}+U\frac{\partial \ln(\zeta U)}{\partial t}+U\frac{\partial U}{\partial x}=-\frac{1}{\rho_0}\frac{\partial P}{\partial x}-\frac{\partial\overline{u^2}}{\partial x}-\frac{\partial\overline{uv}}{\partial y}+\frac{\overline{u^2}-\overline{v^2}}{L_a}+\frac{2\overline{uv}}{R}-g_x\frac{T}{T_0}$$
$$+\nu\left[\frac{\partial^2 U}{\partial x^2}+\frac{\partial^2 U}{\partial y^2}-\frac{2}{L_a}\frac{\partial U}{\partial x}-\frac{1}{R}\frac{\partial U}{\partial y}-\frac{U}{R^2}\right], \quad (4.1)$$

$$\frac{U^2}{R}=-\frac{1}{\rho_0}\frac{\partial P}{\partial y}-\frac{\partial\overline{uv}}{\partial x}-\frac{\partial\overline{v^2}}{\partial y}+\frac{2\overline{uv}}{L_a}+\frac{(\overline{u^2}-\overline{v^2})}{R}-g_y\frac{T}{T_0}$$
$$+\nu\left[-\frac{\partial^2 U}{\partial x\,\partial y}+\frac{1}{L_a}\frac{\partial U}{\partial y}+\frac{\partial}{\partial x}\left(\frac{U}{R}\right)+\frac{U}{RL_a}\right]. \quad (4.2)$$

P is the mean static pressure and T the mean deviation of the temperature from the adiabatic profile T_0. Henceforth we replace the buoyancy terms $g(i)\,\Delta\rho/\rho_0$ by the expressions $-g(i)\,T/T_0$, which are equivalent in the Boussinesq approximation and which have greater currency in turbulence work.

The transport equations for the four non-zero components of Reynolds stress are

$$\frac{\partial\overline{uv}}{\partial t}+\overline{uv}\frac{\partial \ln(\zeta U^2)}{\partial t}+U\frac{\partial\overline{uv}}{\partial x}=-2\overline{u^2}\frac{U}{R}+\overline{v^2}\Omega-\frac{1}{T_0}(g_x\overline{v\theta}+g_y\overline{u\theta})$$
$$-\frac{1}{\rho_0}\left(\overline{v\frac{\partial p}{\partial x}}+\overline{u\frac{\partial p}{\partial y}}\right)-\left(\frac{\partial\overline{u^2v}}{\partial x}+\frac{\partial\overline{uv^2}}{\partial y}\right)+\frac{1}{L_a}(2\overline{u^2v}-\overline{v^3})$$
$$+\frac{1}{R}(2\overline{uv^2}-\overline{u^3})+\nu\left\{\overline{u\nabla^2 v+v\nabla^2 u}+\frac{\partial}{\partial x}\left(\frac{\overline{u^2}-\overline{v^2}}{R}\right)\right.$$
$$\left.-\frac{\partial}{\partial y}\left(\frac{\overline{u^2}-\overline{v^2}}{L_a}\right)-\frac{1}{L_a}\frac{\partial}{\partial x}\overline{uv}-\frac{1}{R}\frac{\partial}{\partial y}\overline{uv}-2\overline{uv}\left(\frac{1}{L_a^2}+\frac{1}{R^2}\right)\right\}, \quad (4.3)$$

where p and θ are the fluctuations in pressure and temperature about P and T respectively, and $\nabla^2 \equiv [\partial^2/\partial x^2+\partial^2/\partial y^2+\partial^2/\partial z^2]$,

$$\frac{\partial\overline{u^2}}{\partial t}+\overline{u^2}\frac{\partial \ln(\zeta^2 U^2)}{\partial t}+U\frac{\partial\overline{u^2}}{\partial t}=-2\overline{u^2}\frac{\partial U}{\partial x}+2\overline{uv}\Omega-2\frac{g_x}{T_0}\overline{u\theta}-\frac{2}{\rho_0}\overline{u\frac{\partial p}{\partial x}}$$
$$-\left(\frac{\partial\overline{u^3}}{\partial x}+\frac{\partial\overline{u^2v}}{\partial y}\right)+\frac{1}{L_a}(\overline{u^3}-2\overline{uv^2})+\frac{3}{R}\overline{u^2v}$$
$$+2\nu\left\{\overline{u\nabla^2 u}+\overline{u\frac{\partial}{\partial y}\left(\frac{v}{L_a}\right)}-\overline{u\frac{\partial}{\partial x}\left(\frac{v}{R}\right)}\right.$$
$$\left.+\overline{\frac{u}{L_a}\left[\frac{\partial v}{\partial y}-\frac{\partial u}{\partial x}\right]}-\overline{\frac{u}{R}\left[\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right]}-\overline{u^2}\left[\frac{1}{L_a^2}+\frac{1}{R^2}\right]\right\}, \quad (4.4)$$

$$\frac{\partial\overline{v^2}}{\partial t}+\overline{v^2}\frac{\partial \ln(U^2)}{\partial t}+U\frac{\partial\overline{v^2}}{\partial x}=-4\overline{uv}\frac{U}{R}+2\overline{v^2}\frac{\partial U}{\partial x}-2\frac{g_y}{T_0}\overline{v\theta}-\frac{2}{\rho_0}\overline{v\frac{\partial p}{\partial y}}$$
$$-\left(\frac{\partial\overline{uv^2}}{\partial x}+\frac{\partial\overline{v^3}}{\partial y}\right)+\frac{3\overline{uv^2}}{L_a}+\frac{1}{R}(\overline{v^3}-2\overline{u^2v})$$

$$+2\nu\left\{\overline{v\nabla^2 v}+\overline{v\frac{\partial}{\partial x}\left(\frac{u}{R}\right)}-\overline{v\frac{\partial}{\partial y}\left(\frac{u}{L_a}\right)}\right.$$

$$\left.-\frac{\overline{v}}{L_a}\left[\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right]+\frac{\overline{v}}{R}\left[\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right]-\overline{v^2}\left[\frac{1}{L_a^2}+\frac{1}{R^2}\right]\right\}, \qquad (4.5)$$

$$\frac{\partial\overline{w^2}}{\partial t}+\frac{U\partial\overline{w^2}}{\partial x}=-\frac{2}{\rho_0}\overline{\frac{w\partial p}{\partial x}}-\left(\frac{\partial\overline{uw^2}}{\partial x}+\frac{\partial\overline{vw^2}}{\partial y}\right)+\frac{1}{L_a}\overline{uw^2}+\frac{1}{R}\overline{vw^2}+2\nu\overline{w\nabla^2 w}. \qquad (4.6)$$

The mean continuity equation, as already stated in (3.13), takes the trivial form

$$\frac{\partial U}{\partial x}-\frac{\partial U}{\partial x}=0,$$

but the continuity condition for the velocity fluctuations is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}-\frac{u}{L_a}-\frac{v}{R}=0. \qquad (4.7)$$

The turbulent kinetic energy $1/2\overline{q^2}$ is one-half the trace of the Reynolds-stress tensor. Its transport equation can be obtained by summing (4.4), (4.5) and (4.6) or, more directly, by the procedure outlined in §3. Since $1/2q^2$ is a scalar and so invariant under a coordinate transformation, its first covariant derivative is simply the partial derivative; as a result it may be derived from the Cartesian equation with considerably less labour than the equations for the individual stresses. It takes the form

$$\frac{\partial}{\partial t}\tfrac{1}{2}\overline{q^2}+\frac{1}{2}\left[\overline{u^2}\frac{\partial}{\partial t}\ln(\zeta^2U^2)+\overline{v^2}\frac{\partial}{\partial t}\ln(U^2)\right]+U\frac{\partial}{\partial x}\tfrac{1}{2}\overline{q^2}$$

$$=-(\overline{u^2}-\overline{v^2})\frac{\partial U}{\partial x}-\overline{uv}\left[\frac{\partial U}{\partial y}+\frac{U}{R}\right]$$

$$-\frac{1}{T_0}[g_x\overline{u\theta}+g_y\overline{v\theta}]-\left[\frac{\partial}{\partial x}\overline{u\left(\frac{p}{\rho_0}+\tfrac{1}{2}q^2\right)}+\frac{\partial}{\partial y}\overline{v\left(\frac{p}{\rho_0}+\tfrac{1}{2}q^2\right)}\right]$$

$$+\frac{1}{L_a}\overline{u\left(\frac{p}{\rho_0}+\tfrac{1}{2}q^2\right)}+\frac{1}{R}\overline{v\left(\frac{p}{\rho_0}+\tfrac{1}{2}q^2\right)}+\nu\left\{\overline{u\nabla^2u}+\overline{v\nabla^2v}+\overline{w\nabla^2w}\right.$$

$$+\frac{2}{L_a}\left[\overline{u\frac{\partial v}{\partial y}}-\overline{v\frac{\partial u}{\partial y}}\right]-\frac{2}{R}\left[\overline{u\frac{\partial v}{\partial x}}-\overline{v\frac{\partial u}{\partial x}}\right]$$

$$\left.-\frac{1}{2L_a}\frac{\partial}{\partial x}[\overline{u^2}+\overline{v^2}]-\frac{1}{2R}\frac{\partial}{\partial y}[\overline{u^2}+\overline{v^2}]-(\overline{u^2}+\overline{v^2})\left[\frac{1}{L_a^2}+\frac{1}{R^2}\right]\right\}. \qquad (4.8)$$

Written in this way the moment equations take the familiar Cartesian forms with simplified advection and the appearance of explicit extra terms in curvature and acceleration in the highest moments. A comparison between the transport equation for a Reynolds stress such as (4.3) and the equation for the same component in (s, n)-coordinates (see e.g. Castro & Bradshaw 1976, p. 286) emphasizes the simplicity of (ϕ, ψ)-coordinates. The appearance of R and L_a in the equations allows a ready appreciation of the relative orders of magnitude of curvature, acceleration and shear effects.

The scales appropriate to mean velocity U and derivatives of mean moments along a streamline are Q and L_a respectively. Gradients of mean moments in the y-direction have a lengthscale l_y, where l_y might be the shear-layer thickness δ or, more generally, the 'mixing length', $l = u_*/\Omega$. $u_* = |\overline{uv}|^{\frac{1}{2}}$ is the friction velocity. Bradshaw's (1973) definition of a 'fairly thin shear layer' (FTSL) is roughly equivalent to assuming that

l/R, $l/L_a < 0.1$. With these definitions we see that the mean streamwise momentum equation expresses a balance between acceleration and pressure gradient along the streamline and the cross-stream gradient of shear stress. (In this and what follows, we ignore the effect of buoyancy forces so as not to complicate the argument; the manner of their inclusion is obvious. For the same reason we restrict the discussion to flows of sufficiently high Reynolds number that viscous effects are not significant in the momentum balance except very close to a smooth wall.) Curvature effects can only be significant if $(2\overline{uv}/R)/(\partial\overline{uv}/\partial y) \approx 1$.

Choosing $l_y = l$, this condition may be rewritten

$$\left(\frac{u_*^2}{U^2}\right)^{\frac{1}{2}} \frac{2U}{R\Omega} \approx 1,$$

where $2U/R\Omega$ is the 'curvature Richardson number' R_c, a measure of the stability of a curved flow to small perturbations. Positive R_c denotes a stable flow (see Bradshaw 1969). In turbulent boundary layers over very rough surfaces $(u_*^2/U^2)^{\frac{1}{2}}$ can be as high as 0.3 (Raupach, Thom & Edwards 1980). In shear layers satisfying the FTSL constraints, however, $|R_c|$ rarely exceeds 0.5. The strongly curved shear layers investigated by Castro & Bradshaw (1976), Gillis & Johnston (1980) and Margolis & Lumley (1965) (representing free shear layers, boundary layers and channel flows) had values of δ/R of 0.1, 0.09, 0.25 respectively, but their corresponding values of R_c were only 0.3, 0.4, 0.3 (I have compared the stable cases). Their values of u_*^2/U^2 were also very small, being at most 8.0×10^3. In rough wall flows or atmospheric surface layers, the velocity gradient, $\partial U/\partial y$, which is the dominant component of Ω close to the surface, varies roughly as u_*, so that the large values of $(u_*/U)^2$ observed over very rough surfaces are generally associated with R_c values, diminished in proportion. It seems that only in the case of extreme distortion of a turbulent flow by a bluff body, where, locally, the FTSL approximations do not hold, would curvature effects become comparable to the 'Bernoulli' terms $(\partial/\partial x)(P+\frac{1}{2}\rho U^2)$.

Applying the same scaling arguments to (4.2) reveals that the balance between mean angular momentum U^2/R, vertical pressure gradient $\partial p/\partial y$ and vertical divergence of normal stress $(\partial/\partial y)\overline{v^2}$ is modified by the curvature terms when the mean-square fluctuations in angular momentum become comparable to U^2/R.

All of the second-moment equations have a similar pattern, the kinetic-energy equation (4.8) being typical. The production term is a generalization of its familiar Cartesian form, normal stresses doing work against normal rates of strain, and shear stresses against the antisymmetric rates of strain. Explicit extra curvature and acceleration terms appear in the highest moments, the transport terms and in the viscous terms. The ratio of the extra curvature and acceleration terms to the leading terms in a FTSL are l_y/R and l_y/L_a. These may in fact be considerable. In the experiment of Castro & Bradshaw with $l_y/R = \delta/R \approx 0.05$, $(\overline{q^2v}/R)_{max}$ attains a value of $0.25[(\partial/\partial y)\overline{q^2v}]_{max}$ when R reaches its minimum value.

Of particular interest are the direct effects of curvature on the pressure-gradient–velocity correlation terms. To see these we first write out the transformed version of the linearized contributions to the Poisson equation for fluctuating pressure. Neglecting viscosity, this is

$$\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}+\frac{\partial^2 p}{\partial z^2}-\frac{1}{L_a}\frac{\partial p}{\partial x}-\frac{1}{R}\frac{\partial p}{\partial y}$$
$$= -2\rho_0\left\{\frac{\partial U}{\partial x}\left[\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\right]+\frac{\partial U}{\partial y}\frac{\partial v}{\partial x}+\frac{1}{L_a}u\frac{\partial U}{\partial x}+\frac{1}{R}\frac{\partial}{\partial y}Uu\right\}+\frac{g_x}{T_0}\frac{\partial \theta}{\partial x}+\frac{g_y}{T_0}\frac{\partial \theta}{\partial y}. \quad (4.9)$$

The left-hand side of (4.9) is the Laplacian in the present coordinates. The Green-function solution to (4.9) is most simply obtained by transforming the solution obtained in Cartesian coordinates (see e.g. Crow 1968). Alternatively, a curvilinear Green function can be derived. Whichever course is chosen, the relative orders of magnitude of the various terms contributing to the final expression $\overline{u_i(\partial p/\partial x_j)}$ reduce to those of the integrand, the right-hand side of (4.9). (This is not strictly true, since the magnitude of the correlations $\overline{u_i u_j}$ has an effect but not large enough to affect our conclusions here.)

In a FTSL the leading term involves the mean shear, that is, $2(\partial U/\partial y)\ \partial v/\partial x$. Relative to this term the explicit curvature term, $(2/R)(\partial/\partial y)\,Uu$, is of order l_y/R while the *explicit* acceleration $(2/L_a)\,u\,\partial U/\partial x$ is only of order $l_y \mathscr{L}/L_a^2$, where $\mathscr{L}$ is a typical integral lengthscale of the turbulence and we have assumed that $\mathscr{L}$ would be the lengthscale of variations in turbulent fluctuations.

In a shear layer $\mathscr{L} \approx l_y$ usually, so that $(2/L_a)\,u\partial U/\partial x$ is about an order of magnitude smaller than $(2/R)(\partial/\partial y)\,Uu$. The 'Cartesian' acceleration term $2(\partial U/\partial x)\,[\partial u/\partial x - \partial v/\partial y]$ is the same order as the *explicit* curvature term, if $L_a \approx R$. The extra term $(1/L_a)\,u\,\partial U/\partial x$ then can be regarded as the difference between contributions to the pressure by acceleration along a streamline and along a fixed rectilinear axis, that is, the effect of actual streamline divergence.

5. Conclusion

The equations of two-dimensional turbulent fluid motion have been transformed into a coordinate frame of streamlines and their orthogonal trajectories. Vectors and tensors in the transformed equation set are referred to an orthonormal vector basis, consisting of the tangent, principal normal and binormal of the streamline. In the transformed equations advective terms are simplified at the expense of acquiring new terms in the highest moments. These involve two lengthscales – the local radius of curvature of the streamline and the 'e-folding' distance of streamwise acceleration – in a form which makes explicit the magnitude of these influences on the flow field.

Some obvious simplifications in interpretation follow from the new equations: Reynolds shear stress $\rho\overline{uv}$, for example, resumes its role as the rate of turbulent momentum transfer in the cross-stream direction, an interpretation which must be abandoned when distorted flows are analysed in Cartesian coordinates. This and related properties have been exploited by Finnigan & Bradley (1983) in their experimental study of neutrally stratified flow past a shelter belt; the equations lent a welcome clarity to the analysis of a complicated situation.

The suggestion was made in §1 that the new equations might simplify analytical or numerical models of complex flows. In this context a few points can be usefully made. First, the removal of singularities from the transform domain by normalizing the vector basis allows boundary conditions to be specified on solid surfaces if so desired.

Secondly, the mean vorticity is an invariant of the transform and appears in the final equations either as Ω or R, these two quantities being connected through (3.15). R can conveniently be regarded as an additional variable of the flow field. A closure problem (other than the familiar turbulent closure problem attendant on Reynolds averaging) is avoided since there exists only one component of mean velocity U, and the equation for cross-stream mean momentum (4.2) is a 'spare' equation.

When the streamline pattern is time-dependent, the integrating factor ζ appears explicitly in the equations. We can use the results of §3 to rewrite the expression (2.11) for ζ in the new coordinate system. In terms of physical quantities it becomes

$$\frac{\zeta}{\zeta_0} = \frac{1}{U} \exp \int \frac{\mathrm{d}y}{R}. \tag{5.1}$$

When the flow field is unsteady we are therefore faced with a set of integro-differential equations; the question whether they will allow simpler specification of unsteady complex flows must await specific applications.

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Appendix

In §3 we derive a term in the momentum equation (3.11) which arises from temporal change of the streamline pattern. This extra term $U(1)\,(\partial/\partial t)\ln(\zeta Q)$ has the form of an extra body force and appears because we have referred the velocities to a framework in general motion. In this appendix we show that the term above (and the corresponding terms in the second-moment equations) have the correct form.

Let $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ form an orthonormal vector basis, not a function of space or time. The three mutually perpendicular, dimensionless vectors $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ form the basis of a rectangular Cartesian coordinate frame. Hence we may write a vector $\boldsymbol{a}$ as

$$\boldsymbol{a} = a^1_{\mathrm{C}}\boldsymbol{i} + a^2_{\mathrm{C}}\boldsymbol{j} + a^3_{\mathrm{C}}\boldsymbol{k}, \tag{A 1}$$

where a^i_{C} are the *components* of $\boldsymbol{a}$ in the Cartesian frame.

We are at perfect liberty to describe $\boldsymbol{a}$ by referring it to any other basis or triplet of linearly independent vectors $\boldsymbol{g}_{(1)}$, $\boldsymbol{g}_{(2)}$, $\boldsymbol{g}_{(3)}$, that is, we could write

$$\boldsymbol{a} = a^1\boldsymbol{g}_{(1)} + a^2\boldsymbol{g}_{(2)} + a^3\boldsymbol{g}_{(3)} = a^i\boldsymbol{g}_{(i)}. \tag{A 2}$$

The $\boldsymbol{g}_{(i)}$ may be functions of space or time, and need not be dimensionless; if they are not, then the components a^i of $\boldsymbol{a}$ have dimensions

$$D(a^i) = \frac{D(\boldsymbol{a})}{D(\boldsymbol{g}_{(i)})}. \tag{A 3}$$

In the transformation to (ϕ, ψ, x^3)-coordinates, we have chosen $g_{(i)}$ by making them (implicitly) the tangent vectors to the coordinate lines at any point. Since the coordinate lines are the intersections of the surfaces $x^i(\boldsymbol{y}) =$ constant, $\boldsymbol{y}$ being the position vector of a point in the Cartesian system ($\boldsymbol{y} = y^1\boldsymbol{i} + y^2\boldsymbol{j} + y^3\boldsymbol{k}$), the three tangent vectors may be written as

$$\boldsymbol{g}_{(1)} = \frac{\partial \boldsymbol{y}}{\partial x^1}, \quad \boldsymbol{g}_{(2)} = \frac{\partial \boldsymbol{y}}{\partial x^2}, \quad \boldsymbol{g}_{(3)} = \frac{\partial \boldsymbol{y}}{\partial x^3}, \tag{A 4}$$

and hence have dimensions:

$$[\boldsymbol{g}_{(1)}] = \frac{T}{L}, \quad [\boldsymbol{g}_{(2)}] = \frac{T}{L}, \quad [\boldsymbol{g}_{(3)}] = 0. \tag{A 5}$$

Now we know that the following equation of motion holds in the fixed Cartesian framework:

$$\frac{\partial V^i}{\partial t} + V^j \frac{\partial V^i}{\partial y^j} = -\frac{1}{\rho}\frac{\partial P}{\partial y^i} + \nu \frac{\partial^2 V^i}{\partial y^j \partial y^j}. \tag{A 6}$$

Let us rewrite the left-hand side, the total acceleration, using

$$\boldsymbol{V} = V^1\boldsymbol{i} + V^2\boldsymbol{j} + V^3\boldsymbol{k} = U^1\boldsymbol{g}_{(1)} + U^2\boldsymbol{g}_{(2)} + U^3\boldsymbol{g}_{(3)} = U^i\boldsymbol{g}_{(i)}, \tag{A 7}$$

then

$$\frac{\partial V^i}{\partial t} + V^j\frac{\partial V^i}{\partial y^j} = \frac{\partial}{\partial t}(U^i\boldsymbol{g}_{(1)}) + V^j\frac{\partial}{\partial y^j}(U^i\boldsymbol{g}_{(i)}) \equiv \frac{\partial}{\partial t}(U^i\boldsymbol{g}_{(i)}) + \left[\frac{\partial y^j}{\partial x^k}U^k\right]\frac{\partial x^k}{\partial y^j}\frac{\partial}{\partial x^k}(U^i\boldsymbol{g}_{(i)})$$
$$\equiv \frac{\partial}{\partial t}(U^i\boldsymbol{g}_{(i)}) + \boldsymbol{g}_{(k)}U^k\frac{1}{\boldsymbol{g}_{(k)}}\frac{\partial}{\partial x^k}(U^i\boldsymbol{g}_{(i)}). \tag{A 8}$$

In the expressions above we have replaced

$$\boldsymbol{g}_{(i)} = \frac{\partial \boldsymbol{y}}{\partial x^i} \quad \text{by} \quad \boldsymbol{g}_{(i)} = \frac{\partial y^j}{\partial x^i}.$$

This is possible because the basis $(\boldsymbol{i},\boldsymbol{j},\boldsymbol{k})$ is both orthonormal and not a function of space, so that

$$\frac{\partial \boldsymbol{i}}{\partial x^i} = \frac{\partial \boldsymbol{j}}{\partial x^i} = \frac{\partial \boldsymbol{k}}{\partial x^i} = 0.$$

(in a rectangular Cartesian framework, vectors are represented by their components without ambiguity!)

Rearranging (A 8) we get:

$$\boldsymbol{g}_{(i)}\frac{\partial U^i}{\partial t} + U^i\frac{\partial \boldsymbol{g}_{(i)}}{\partial t} + U^k\left[\boldsymbol{g}_{(i)}\frac{\partial U^i}{\partial x^k} + U^i\frac{\partial \boldsymbol{g}_{(i)}}{\partial x^k}\right]; \tag{A 9}$$

however,

$$U^i\frac{\partial \boldsymbol{g}_{(i)}}{\partial x^k} = U^i\Gamma^j_{ik}\boldsymbol{g}_{(j)}$$

(Aris 1962, p. 164) and by interchanging the dummy indices

$$U^i\frac{\partial \boldsymbol{g}_{(i)}}{\partial x^k} = U^j\Gamma^i_{jk}\boldsymbol{g}_{(i)}, \tag{A 10}$$

so that

$$\frac{\partial V^i}{\partial t} + V^j\frac{\partial V^i}{\partial y^j} = \boldsymbol{g}_{(i)}\left[\frac{\partial U^i}{\partial t} + \frac{U^i}{\boldsymbol{g}_{(i)}}\frac{\partial}{\partial t}\boldsymbol{g}_{(i)}\right] + \boldsymbol{g}_{(i)}U^k\left[\frac{\partial U^i}{\partial x^k} + \Gamma^i_{jk}U^j\right].$$

However, the last term in the square brackets is just the covariant derivative defined in (3.5), so

$$\frac{\partial V^i}{\partial t} + V^j\frac{\partial V^i}{\partial y^i} = \boldsymbol{g}_{(i)}\left\{\frac{\partial U^i}{\partial t} + U^i\frac{\partial}{\partial t}\ln \boldsymbol{g}_{(i)} + U^kU^i_{,k}\right\}. \tag{A 11}$$

It is clear that the extra terms $U^i(\partial/\partial t)\ln \boldsymbol{g}_{(i)}$ and $U^j\Gamma^i_{jk}$ arise entirely from the variation in time and space of the base vectors $\boldsymbol{g}_{(i)}$.

Now it can be shown that

$$g_{ij} = \boldsymbol{g}_{(i)}\cdot\boldsymbol{g}_{(j)}$$

and for orthogonal coordinate lines, $g_{ij} = 0$ $(i \neq j)$, so that

$$g_{ii} = \boldsymbol{g}^2_{(i)},$$
$$|\boldsymbol{g}_{(i)}| = (g_{ii})^{\frac{1}{2}} \quad \text{(no summation)}. \tag{A 12}$$

If we compare (A 11) and (A 12) with the definition of a physical component (3.10), we see that the appearance of $g_{(i)}$ as a factor of the right-hand side of (A 11) ensures that the equation is dimensionally homogeneous; it also serves to clarify the meaning of physical components.

The analysis outlined above works equally well for the right-hand side of (A 6) and provides the justification for step (2) of the procedure of equation transformation outlined in §3.

If the $\boldsymbol{g}_{(i)}$ had been chosen to be a triad of orthonormal vectors, rotating with constant angular velocity ω^j relative to $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$, then $\Gamma^i_{jk} = 0$, and the only extra term in (A 9) would be $U^i(\partial \boldsymbol{g}_{(i)}/\partial t)$. For steady rotation, however,

$$\frac{\partial \boldsymbol{g}_{(i)}}{\partial t} = \epsilon_{ijk}\omega^j \boldsymbol{g}_{(p)} g^{pk}, \qquad \text{(A 13)}$$

where the metric $g^{pk} = \text{diag}\,(1, 1, 1)$ and the label (p) on $g_{(p)}$ can be regarded as a covariant index for the purposes of summation, and

$$U^i \frac{\partial \boldsymbol{g}_{(i)}}{\partial t} = \epsilon_{ijk}\omega^i(U^k g^{pk} \boldsymbol{g}_{(p)}). \qquad \text{(A 14}a\text{)}$$

In the notation of Cartesian tensors, where the distinction between co- and contravariance is not made, (A 14) is simply

$$U_i \frac{\partial \boldsymbol{g}_{(i)}}{\partial t} = \epsilon_{ijk}\omega^i U^k. \qquad \text{(A 14}b\text{)}$$

The reader will recognize the right-hand side of (A 14*b*) as the familiar expression for the Coriolis force.

Rather than combine the transformation to (ϕ, ψ, x^3)-coordinates with any rigid-body motion of the entire transform domain (that is, essentially, of the flow boundaries) into a single metric, it is less confusing to split the transformation into two parts. First, we transform from fixed Cartesian axes to Cartesian axes in rigid motion relative to them. Extra body forces caused by Coriolis effects or rectilinear acceleration will then appear explicitly in a Cartesian equation of motion. Secondly, we transform this equation to streamline co-ordinates.

Motion of the streamlines relative to the fixed boundaries of the flow will then result in a term (in the momentum equation)

$$F^i = U^i(\partial/\partial t)\ln(g_{ii})^{\frac{1}{2}} \quad \text{(no summation)}. \qquad \text{(A 15)}$$

There are three possibilities:

$$(a) \quad i = 1, \quad (g_{ii})^{\frac{1}{2}} = \frac{1}{\zeta Q};$$

$$(b) \quad i = 2, \quad (g_{ii})^{\frac{1}{2}} = \frac{1}{Q};$$

$$(c) \quad i = 3, \quad (g_{ii})^{\frac{1}{2}} = 1.$$

Consider the first case, $i = 1$:

$$F^i = U^i \zeta Q \frac{\partial}{\partial t}\frac{1}{\zeta Q} = -U^i\left[\frac{\partial}{\partial t}\ln\zeta + \frac{\partial}{\partial t}\ln Q\right], \qquad \text{(A 16)}$$

but

$$\frac{\partial}{\partial t}\ln\zeta = \frac{\partial}{\partial t}\oint\frac{\Omega}{Q^2}\,dx^2 \qquad \text{(A 17)}$$

Obviously, in this case, knowledge of the global properties of the velocity field is necessary to compute (A 17). Physically, this is because the behaviour of a streamline at some point is determined by the *total* mass flow through the line connecting that point with a solid boundary.

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